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# Existence of Solutions with Moving Singularities for a Semilinear Parabolic Equation

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## Abstract

We study the Cauchy problem for a semilinear parabolic equation with a power nonlinearity. It is known that in some parameter range, the equation has a singular steady state. Our concern is a solution with a moving singularity that is obtained by perturbing the singular steady state. By the formal expansion, it turns out that the correction term must satisfy the heat equation with inverse-square potential near the singular point. From the well-posedness of this equation, we see that there appears a critical exponent. Paying attention to this exponent, given a motion of the singular point and suitable initial data, we establish the time-local existence result.

## 1 Introduction

We study singular solutions of the semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + u^p & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.1)$$

where  $p > 1$  is a parameter and  $u_0 \in L^1_{loc}(\mathbb{R}^N)$  is a nonnegative function. It is known that for

$$N \geq 3, \quad p > p_{sing} := \frac{N}{N-2},$$

(1.1) has an explicit singular steady state  $\varphi(|x|) \in C^\infty(\mathbb{R}^N \setminus \{0\})$  with a singular point 0;

$$\varphi(|x|) = L|x|^{-m}, \quad m = \frac{2}{p-1}, \quad L^{p-1} = m(N-m-2).$$

Then  $\varphi(|x|)$  satisfies (1.1) in the distribution sense, and

$$\varphi_{rr} + \frac{N-1}{r} \varphi_r + \varphi^p = 0, \quad r = |x| > 0. \quad (1.2)$$

Clearly, the spatial singularity of  $u = \varphi(|x|)$  persists for all  $t > 0$ , but the singular point does not move in time.

Our aim of this paper is to discuss the existence of a solution of (1.1) whose spatial singularity moves in time. More precisely, we define a solution with a moving singularity as follows.

**Definition 1.** The function  $u(x, t)$  is said to be a solution of (1.1) with a moving singularity  $\xi(t) \in \mathbb{R}^N$  for  $t \in (0, T)$ , where  $0 < T \leq \infty$ , if the following conditions hold:

- (i)  $u, u^p \in C([0, T]; L^1_{loc}(\mathbb{R}^N))$  satisfy (1.1) in the distribution sense.
- (ii)  $u(x, t)$  is defined on  $\{(x, t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\}, t \in (0, T)\}$ , and is twice continuously differentiable with respect to  $x$  and continuously differentiable with respect to  $t$ .
- (iii)  $u(x, t) \rightarrow \infty$  as  $x \rightarrow \xi(t)$  for every  $t \in [0, T)$ .

In this paper, we study the time-local existence for a solution with a moving singularity of the Cauchy problem (1.1). In order to state our result, we first introduce a critical exponent given by

$$p_* := \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}},$$

which appeared in the papers of Véron [8] and Chen-Lin [3]. It was shown in [8] that  $p_*$  is related to the linearized stability of the singular steady state, while it was shown in [3] that  $p_*$  plays a crucial role for the existence of solutions with a prescribed singular set of the Dirichlet problem

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^N$ . In fact, in [3], they proved that if  $N \geq 3$ ,  $p_{sing} < p < p_*$ , then for any closed set  $K \subset \Omega$ , there exists a singular solution having  $K$  as a singular set. We note that  $p_*$  is larger than  $p_{sing}$  and is smaller than the Sobolev critical exponent  $p_S := (N+2)/(N-2)$ . We also introduce the important numbers

$$\begin{aligned} \lambda_1 &:= \frac{N-2 - \sqrt{(N-2)^2 - 4pL^{p-1}}}{2}, \\ \lambda_2 &:= \frac{N-2 + \sqrt{(N-2)^2 - 4pL^{p-1}}}{2}. \end{aligned}$$

We note that for  $N \geq 3$ ,  $p_{sing} < p < p_*$ , the constants  $\lambda_1 < \lambda_2$  are positive roots of

$$\lambda^2 - (N-2)\lambda + pI^{p-1} = 0.$$

Finally, for  $a \in \mathbb{R}$ ,  $[a]$  denotes the largest integer not greater than  $a$ .

Our result is concerning the time-local existence of a solution of (1.1) with a moving singularity.

**Theorem 1.** *Let  $N \geq 3$  and  $p_{sing} < p < p_*$ . Assume the following conditions:*

- (A1)  $\xi(t) \in C^{i+\alpha}([0, \infty); \mathbb{R}^N)$  ( $\alpha > 0$ ) with  $i = [\frac{m-\lambda_2}{2}] + 1$ .
- (A2)  $u_0$  is nonnegative and continuous in  $x \in \mathbb{R}^N \setminus \xi(0)$ , and is uniformly bounded for  $|x - \xi(0)| \geq 1$ .
- (A3) If  $m - \lambda_2$  is not an integer, then

$$u_0(x) = L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^{[m-\lambda_2]} b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i + O(|x - \xi(0)|^{m-\lambda_2+\varepsilon}) \right\}$$

as  $x \rightarrow \xi(0)$  for some  $\varepsilon > 0$ , where  $b_i(\omega, t)$  are functions on  $S^{N-1}$  defined later by (2.3)-(2.5). If  $m - \lambda_2$  is an integer, then

$$u_0(x) = L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^{m-\lambda_2} b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i + c(0)|x - \xi(0)|^{m-\lambda_2} \log |x - \xi(0)| + O(|x - \xi(0)|^{m-\lambda_2+\varepsilon}) \right\}$$

as  $x \rightarrow \xi(0)$  for some  $\varepsilon > 0$ , where  $b_i(\omega, t)$  are functions on  $S^{N-1}$  defined later by (2.3)-(2.5) and  $b_{m-\lambda_2}(\omega, t)$  and  $c(t)$  satisfy (3.1)

Then for some  $T > 0$ , there exists a solution of (1.1) with a moving singularity  $\xi(t)$ .

**Remark 1.** If  $N \geq 3$  and

$$p_{sing} < p < \min \left\{ p_*, \frac{3N+5}{3N-3} \right\},$$

then  $0 \leq m - \lambda_2 < 1$  so that  $[m - \lambda_2] = 0$ . In this case, (A1) implies  $\xi(t) \in C^{1+\alpha}([0, \infty); \mathbb{R}^N)$  ( $\alpha > 0$ ), and (A3) is simplified as

$$u_0(x) = L|x - \xi(0)|^{-m} + O(|x - \xi(0)|^{-\lambda_2+\varepsilon}) \quad \text{as } x \rightarrow \xi(0). \quad (1.3)$$

In this paper, we consider only the time-local existence of the Cauchy problem with a moving singularity. Needless to say, the existence of time-global solutions are important questions. Also, when the solution with a moving singularity is not time-global, it is interesting to ask what happens at the maximal existence time. These questions will be future works.

This paper is organized as follows: In Section 2 we carry out formal analysis for a solution of (1.1) as a perturbation of the singular steady state. In Section 3 we state the outline of proof of the time-local existence.

## 2 Formal expansion at a singular point

In this section, we consider the formal expansion of a solution  $u(x, t)$  of (1.1) with a moving singularity  $\xi(t)$ . Assuming that the solution resembles the singular steady state around  $\xi(t)$ , we may naturally expand  $u(x, t)$  as

$$u(x, t) = Lr^{-m} \left\{ 1 + \sum_{i=1}^k b_i(\omega, t)r^i + v(y, t)r^m \right\}, \quad (2.1)$$

where

$$y = x - \xi(t), \quad r = |x - \xi(t)|, \quad \omega = \frac{1}{r}(x - \xi) \in S^{N-1}, \quad k = [m],$$

and the remainder term  $v$  satisfies

$$v(y, t) = o(|y|^{-m}) \quad \text{as } |y| \rightarrow 0. \quad (2.2)$$

Substituting (2.1) into (1.1), and using

$$r_t = -\frac{(x - \xi) \cdot \xi_t}{r}, \quad \omega_t = -\frac{1}{r}\xi_t + \frac{\omega \cdot \xi_t}{r}\omega,$$

$$\Delta = \partial_{rr} + \frac{N-1}{r}\partial_r + \frac{1}{r^2}\Delta_{S^{N-1}}$$

and the Taylor expansion, we compare the coefficients of  $r^{-m+i-2}$  for  $i = 0, 1, \dots, k$ . Then we obtain

$$r^{-m-2}; (Lr^{-m})_{rr} + \frac{N-1}{r}(Lr^{-m})_r + (Lr^{-m})^p = 0,$$

$$r^{-m-1}; \Delta_{S^{N-1}}b_1 + \{(-m+1)(N-m-1) + pm(N-m-2)\}b_1 = m\omega \cdot \xi_t, \quad (2.3)$$

$$r^{-m}; \Delta_{S^{N-1}} b_2 + \{(-m+2)(N-m) + pm(N-m-2)\} b_2 \\ = (m-1)b_1 \omega \cdot \xi_t - (\xi_t - (\omega \cdot \xi_t)\omega) \cdot \nabla b_1 + \frac{p(p-1)}{2} m(N-m-2)b_1^2, \quad (2.4)$$

$$r^{-m+i-2}; \Delta_{S^{N-1}} b_i + \{(-m+i)(N-m+i-2) + pm(N-m-2)\} b_i \\ = G_i(\omega; b_1, b_2, \dots, b_{i-1}, \xi) \quad (i = 3, 4, \dots, k). \quad (2.5)$$

where  $\Delta_{S^{N-1}}$  is the Laplace-Beltrami operator on  $S^{N-1}$  and the function  $G_i(\omega; b_1, b_2, \dots, b_{i-1}, \xi)$  on  $S^{N-1} \times [0, \infty)$  is determined by  $(b_1, b_2, \dots, b_{i-1}, \xi)$ .

The equality for  $r^{-m-2}$  always holds by (1.2). From other equations, we have the above system of inhomogeneous elliptic equations for  $b_i$  on  $S^{N-1}$ : By these equations,  $b_1, b_2, \dots$  are determined sequentially.

Let us consider the solvability of (2.3), (2.4) and (2.5). It is well known (see, e.g. [2]) that for every  $j = 0, 1, 2, \dots$ , the eigenvalues of  $-\Delta_{S^{N-1}}$  are given by

$$\mu_j = j(N+j-2), \quad j = 0, 1, 2, \dots$$

and the eigenspace  $E_j$  associated with  $\mu_j$  is given by

$$E_j = \{f|_{S^{N-1}} : f \text{ is a harmonic homogeneous polynomial of degree } j\}.$$

Therefore, unless

$$(-m+i)(N-m+i-2) + pm(N-m-2) = j(N+j-2), \quad (2.6)$$

the operators in the left-hand side of (2.3), (2.4) and (2.5) are invertible. We define a set  $\Lambda$  by

$$\Lambda := \left\{ p > 1 : (2.6) \text{ holds for some } i \in \{1, 2, \dots, [\frac{2}{p-1}]\}, j \in \{0, 1, 2, \dots, i\} \right\}.$$

Moreover, we consider  $G_i(\omega; b_1, b_2, \dots, b_{i-1}, \xi)$  in detail and obtain next lemma.

**Lemma 1.** *Suppose that  $\xi(t)$  satisfies (A1). If  $p \notin \Lambda$ , then there exist  $b_1(\omega, t), b_2(\omega, t), \dots, b_k(\omega, t) \in C^{\infty,1}(S^{N-1} \times [0, \infty))$  such that (2.3), (2.4) and (2.5) hold.*

By this lemma, in order to consider the existence of the solution of (1.1) with a moving singularity, it suffices to consider  $v(y, t)$ . By taking  $b_i(\omega, t)$  as Lemma 1, (1.1) is satisfied if  $v(y, t)$  satisfies

$$v_t = \Delta v + \xi_t \cdot \nabla v + F(v, y, t) \quad \text{in } \mathbb{R}^N \times (0, \infty). \quad (2.7)$$

where  $F(v, y, t)$  is determined by  $b_1, b_2, \dots, b_k$  and  $\xi$ . After tedious computations, we notice that

$$F(v, y, t) = \frac{pL^{p-1}}{r^2}v + o(r^{-2}) \quad \text{as } r \rightarrow 0.$$

In order to consider the existence of solutions of (2.7), we first consider

$$v_t = \Delta v + \frac{pL^{p-1}}{r^2}v \quad \text{in } \mathbb{R}^N \times (0, \infty). \quad (2.8)$$

This equation has been investigated in [1, 7, 6], and it was shown that (2.8) is well-posed when

$$0 < pL^{p-1} < \frac{(N-2)^2}{4}, \quad (2.9)$$

and

$$|v(y, 0)| \leq Cr^{-\lambda} \quad \text{for some } \lambda_1 < \lambda < \lambda_2, \quad C > 0.$$

The inequalities (2.9) hold if and only if  $p$  satisfies

$$p_{sing} < p < p_* \quad \text{for } N \geq 3, \text{ or } p > p_{JL} := \frac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}} \quad \text{for } N > 10.$$

Here the exponent  $p_{JL}$  was first introduced by Joseph-Lundgren [4] and is known to play an important role for the dynamics of solutions of (1.1).

Since the gradient term in (2.7) and the higher order term of  $F$  do not affect the well-posedness, we must assume (2.9) for the solvability of (2.7). If  $p > p_{JL}$ , then  $\lambda_1 < m$  does not hold so that (2.2) may not be true. Hence we exclude the case  $p_{JL} < p$ . Based on the above formal analysis, we will focus on the case  $p_{sing} < p < p_*$ .

### 3 Time-local existence

Taking into account of the formal analysis in the previous section, we will show the existence of a time-local solution with a moving singularity. To this end, we develop the idea of Marchi [6] for the well-posedness of the linear equation (2.8).

The outline of the proof is divided into three steps. Roughly speaking, we construct a suitable supersolution and subsolution with a moving singularity in Subsection 3.1. In Subsection 3.2, we construct a sequence of approximate solutions and find a convergent subsequence. In Subsection 3.3, we show that the limiting function is indeed a solution of (1.1) with a moving singularity.

### 3.1 Construction of a supersolution and a subsolution

In this subsection, we construct a supersolution and a subsolution of (1.1) that are suitable for our purpose.

First we note that if  $m - \lambda_2$  is not an integer, then (2.6) does not hold for all  $i = 1, 2, \dots, [m - \lambda_2]$ ,  $j = 0, 1, \dots, i$ . Indeed, if (2.6) does not hold for some  $1 \leq i \leq m - \lambda_2$ ,  $j = 1, \dots, i$ , then  $i = -\lambda_2$ ,  $j = 0$ , contradicting that  $m - \lambda_2$  is not an integer. Therefore, if  $m - \lambda_2$  is not an integer, then by Lemma 1 and (A1), we can determine  $b_1(\omega, t)$ ,  $b_2(\omega, t), \dots, b_{[m-\lambda_2]}(\omega, t) \in C^{2,1}(S^{N-1} \times [0, \infty))$  by (2.3), (2.4) and (2.5).

On the other hand, if  $m - \lambda_2$  is an integer, (2.6) holds for  $i = m - \lambda_2$ ,  $j = 0$ . However, we carry out similar argument by replacing  $b_{[m-\lambda_2]}(\omega, t)r^{[m-\lambda_2]}$  with  $(b_{m-\lambda_2}(\omega, t) + c(t) \log r)r^{m-\lambda_2}$  that satisfies

$$\Delta_{S^{N-1}} b_{m-\lambda_2} = (I - P_0)G(\omega, t), \quad c(t) = (N - 2\lambda_2 - 2)^{-1} P_0 G(\omega, t), \quad (3.1)$$

where  $P_0$  is define the projection on  $E_0$  and  $G(\omega, t)$  is the right-hand side of (2.5) with  $i = m - \lambda_2$ .

Now we fix  $\lambda = \lambda_2 - \epsilon$  satisfying

$$\min\{\lambda_1, m - [m - \lambda_2] - 1\} < \lambda < \lambda_2$$

and replace  $k$  defined in Section 2 with  $k := [m - \lambda_2]$ . From (A2) and (A3), it follows that  $u_0 \in C(\mathbb{R}^N \setminus \xi(0)) \cap L^\infty(\mathbb{R}^N \setminus B(\xi(0), 1))$ ,  $u_0 \geq 0$ , and

$$\begin{aligned} u_0(x) = L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^k b_i \left( \frac{x - \xi(0)}{|x - \xi(0)|}, 0 \right) |x - \xi(0)|^i \right. \\ \left. + O(|x - \xi(0)|^{m-\lambda}) \right\} \quad \text{as } x \rightarrow \xi(0). \end{aligned}$$

Then there exist constants  $C > 0$  and  $R > 0$  such that

$$\begin{aligned} \left| u_0(x) - L|x - \xi(0)|^{-m} \left\{ 1 + \sum_{i=1}^k b_i(\omega, 0) \left( \frac{x - \xi(0)}{|x - \xi(0)|} \right) |x - \xi(0)|^i \right\} \right| \\ < CL|x - \xi(0)|^{-\lambda} \quad \text{in } B(\xi(0), R). \end{aligned}$$

Fix any  $T_1 > 0$ .

First we construct a supersolution and a subsolution of (1.1) in a neighborhood of  $\xi(t)$  by using (2.7). By (2.1), we have

$$u_t - \Delta u - u^p = L\{v_t - \Delta v - \xi_t \cdot \nabla v - F(v, y, t)\}.$$



Hence

$$\bar{u}(x, t) = Lr^{-m} \left\{ 1 + \sum_{i=1}^k b_i(\omega, t)r^i + v^+(y, t)r^m \right\}$$

is a supersolution of (1.1) if and only if  $v^+$  is a supersolution of (2.7). Since it follows from tedious calculation that  $\bar{v} := Cr^{-\lambda}$  is a supersolution of (2.7) on  $B_R \times (0, T_1)$  if  $R > 0$  is sufficiently small,

$$\bar{u} := L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^k b_i(\omega, t)|x - \xi(t)|^i + C|x - \xi(t)|^{m-\lambda} \right\}$$

is a supersolution of (1.1) on  $\bigcup_{0 \leq t \leq T_1} B_R(\xi(t)) \times \{t\}$  for small  $R > 0$ . Similarly, we can show that

$$\underline{u} := L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^k b_i(\omega, t)|x - \xi(t)|^i - C|x - \xi(t)|^{m-\lambda} \right\}$$

is a subsolution of (1.1) on  $\bigcup_{0 \leq t \leq T_1} B_R(\xi(t)) \times \{t\}$  for small  $R > 0$ .

Next, we construct a supersolution and a subsolution near infinity. By direct calculation, it is shown that

$$\bar{u} := C_1 \left( 1 - \frac{t}{2T_2} \right)^{-\frac{1}{2(p-1)}}$$

is a supersolution of (1.1) on  $\mathbb{R}^N \setminus B(\xi(t), 1) \times (0, T_2)$ , provided that

$$C_1 > \|u_0\|_{L^\infty(\mathbb{R}^N \setminus B(\xi(0), 1))}, \quad T_2 < 2\sqrt{2}(p-1)C_1^{p-1}.$$

Clearly  $u \equiv 0$  is a subsolution (1.1).

Finally, connecting these supersolutions and subsolutions in the intermediate region, we obtain a supersolution  $\bar{u}$  and a subsolution  $\underline{u}$  such that  $\bar{u}, \bar{u}^p, \underline{u}, \underline{u}^p \in L^1_{loc}(\mathbb{R}^N \times [0, T])$  and the following properties hold:

- (i)  $\bar{u}(x, t)$  and  $\underline{u}(x, t)$  are defined on  $\{(x, t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N \setminus \{\xi(t)\}, t \in [0, T]\}$  and are twice continuously differentiable with respect to  $x$  and continuously differentiable with respect to  $t$ .
- (ii) For every  $t \in [0, T]$ ,  $\bar{u}(x, t), \underline{u}(x, t) \rightarrow \infty$  as  $x \rightarrow \xi(t)$ . In particular,

$$\begin{aligned} \bar{u}(x, t) &= L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^k b_i(\omega, t)|x - \xi(t)|^i + C|x - \xi(t)|^{m-\lambda} \right\}, \\ \underline{u}(x, t) &= L|x - \xi(t)|^{-m} \left\{ 1 + \sum_{i=1}^k b_i(\omega, t)|x - \xi(t)|^i - C|x - \xi(t)|^{m-\lambda} \right\} \end{aligned}$$

for  $|x - \xi(t)| \leq R_0$  and  $0 \leq t \leq T$ .

(iii) The inequalities

$$\begin{aligned} \bar{u}(x, 0) &> u_0(x) > \underline{u}(x, 0) \quad \text{in } \mathbb{R}^N \setminus \{\xi(0)\}, \\ \bar{u}(x, t) &> \underline{u}(x, t) \quad \text{in } \mathbb{R}^N \times [0, T] \setminus \bigcup_{0 \leq t \leq T} (\xi(t), t) \end{aligned}$$

hold.

(iv) The inequalities

$$\begin{aligned} \bar{u}_t &\geq \Delta \bar{u} + \bar{u}^p \quad \text{in } \mathbb{R}^N \times [0, T] \setminus \bigcup_{0 \leq t \leq T} (\xi(t), t), \\ \underline{u}_t &\leq \Delta \underline{u} + \underline{u}^p \quad \text{in } \mathbb{R}^N \times [0, T] \setminus \bigcup_{0 \leq t \leq T} (\xi(t), t) \end{aligned}$$

hold.

for some small  $R_0$  and  $T$ .

### 3.2 Construction of approximate solutions

In this subsection, by using the supersolution and subsolution given in the previous subsection, we construct a series of approximate solutions that is convergent in an appropriate function space.

Define a sequence of bounded domains

$$A_n(t) := \{x \in \mathbb{R}^N : |x - \xi(t)| \leq n, |x - \xi(t)| \geq \frac{1}{n}\} \quad (n = 1, 2, \dots).$$

For each  $n$ , let  $u_n(x, t)$  be a classical solution of

$$\begin{cases} u_{n,t} = \Delta u_n + u_n^p & \text{in } \bigcup_{0 \leq t \leq T} A_n(t) \times \{t\}, \\ u_n = \underline{u} & \text{on } \bigcup_{0 \leq t \leq T} \partial A_n(t) \times \{t\}, \\ u_n(x, 0) = u_{0,n}(x) & \text{in } A_n(0), \end{cases}$$

where the initial value is assumed to satisfy

$$\begin{aligned} \underline{u}(x, 0) &\leq u_{0,n}(x) \leq u_{0,n+1}(x) \leq \bar{u}(x, 0) \quad \text{in } A_n(0), \\ u_{0,n}(x) &= \underline{u}(x, 0) \quad \text{on } \partial A_n(0), u_{0,n} \nearrow u_0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

It is easily seen that  $\underline{u} \leq u_n \leq \bar{u}$  in  $\bigcup_{0 \leq t \leq T} A_n(t) \times \{t\}$  by the comparison principle. Furthermore, by the standard parabolic theory [5] and the Ascoli-Arzelà theorem, from  $\{u_n\}$ , we can obtain a subsequence  $\{u_{n(j)}\}_j$  and some function  $u(x, t)$  such that

$$u_{n(j)} \rightarrow u \text{ locally uniformly in } \mathbb{R}^N \times (0, T) \setminus \bigcup_{0 < t < T} (\xi(t), t) \text{ as } n(j) \rightarrow \infty$$

Hence the limiting function  $u(x, t)$  satisfies

$$\begin{aligned} u &\in C(\mathbb{R}^N \times (0, T) \setminus \bigcup_{0 < t < T} (\xi(t), t)), \\ \underline{u} &\leq u \leq \bar{u} \text{ in } \mathbb{R}^N \times (0, T) \setminus \bigcup_{0 < t < T} (\xi(t), t). \end{aligned}$$

### 3.3 Completion of the proof

In this subsection, we show that the limiting function  $u(x, t)$  obtained in Subsection 3.2 is indeed a solution of (1.1) with a moving singularity  $\xi(t)$  for  $t \in (0, T)$ .

First, by  $\underline{u} \leq u \leq \bar{u}$  and the Lebesgue convergence theorem, we can show that the function  $u$  satisfies (1.1) in the distribution sense. Next, by  $\underline{u} \leq u \leq \bar{u}$  and the standard parabolic theory [5], the function  $u$  has the desired properties as stated in Definition 1. Consequently, it is shown that the function  $u$  is a solution of (1.1) with a moving singularity  $\xi(t)$  for  $t \in (0, T)$ . ■

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